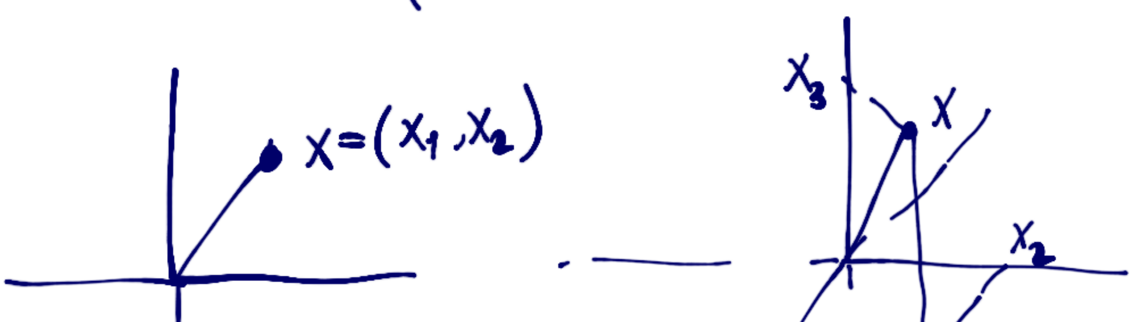
Euclidean space

Basic concepts

we define the Euclidean space $\mathbb{R}^N$, $N \geq 1$ using cartesian coordinates and geometrical concepts.

Each element $\boxed{(x_1, \dots, x_N)}$ of $\mathbb{R}^N$

$$x = (x_1, \dots, x_N)$$



$$x = \sum_{i=1}^{N} x_i e_i \quad , \quad x_i \in \mathbb{R}$$

In particular in $\mathbb{R}^3$, $B = \{ i, j, k \}$

<u>Properties</u> $\lambda, \mu \in \mathbb{R}$, $x, y \in \mathbb{R}^N$

Linear space.

a) Addition of vectors.

$$(x_1, \dots x_N) + (y_1, \dots y_N) = (x_1 + y_1, \dots x_N + y_N)$$

b) Multiplication by scalars, $\lambda \in \mathbb{R}$

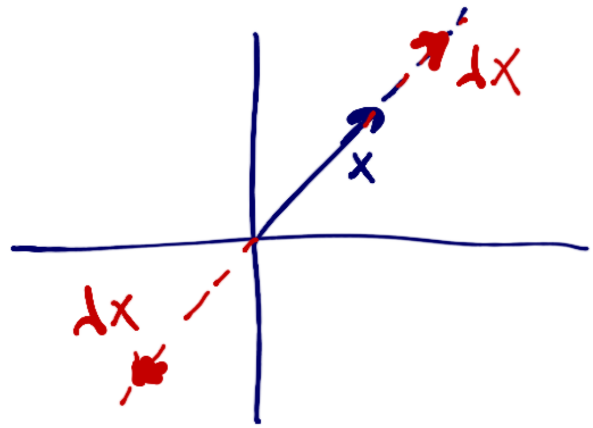
$$\lambda (x_1, \dots x_N) = (\lambda x_1, \dots, \lambda x_N)$$

c) $(\mu \lambda)(x_1, \dots x_N) = \mu \left((\lambda x_1, \dots \lambda x_N) \right)$

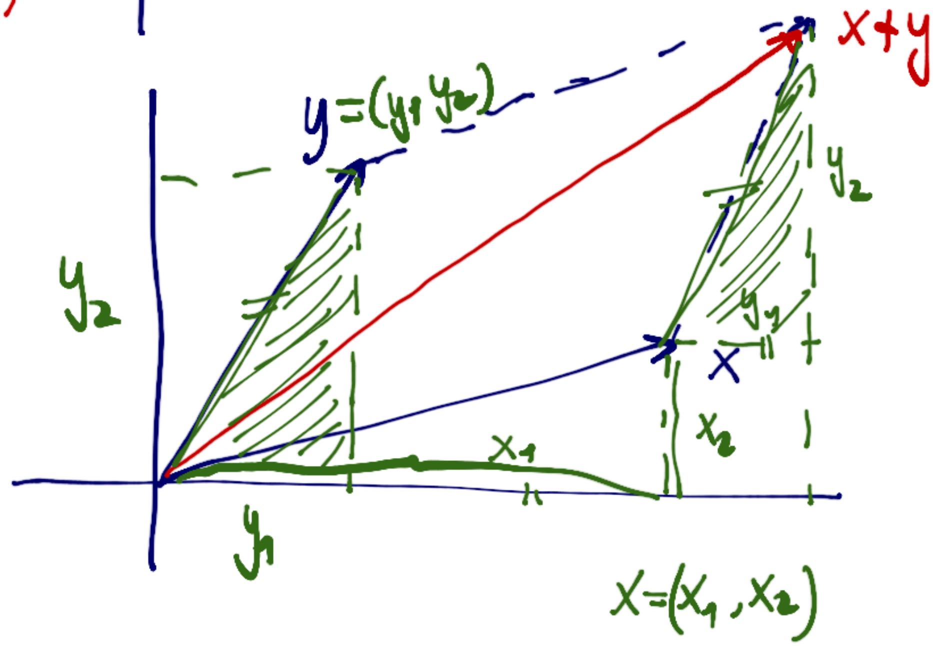
d) $(\mu + \lambda) x = \mu x + \lambda x$ | f) $0 \cdot x = 0 \in \mathbb{R}^N$

Geometrically

λx means a rescaling of the vector x

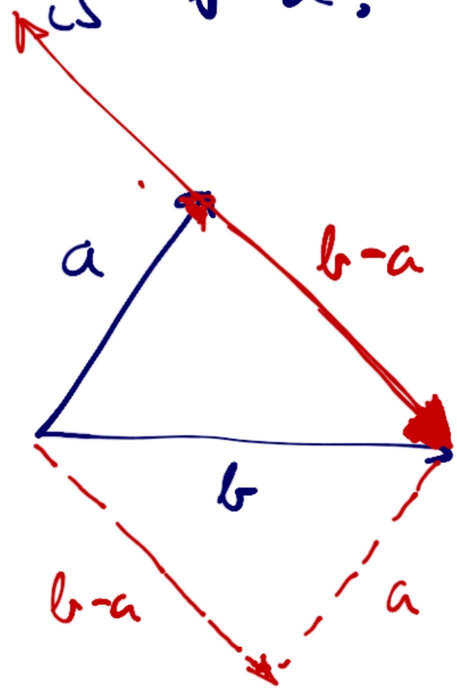


$x+y$



Remark

What is $b-a$? $a, b \in \mathbb{R}^N$



analytically

$\underline{a + (b-a) = b}$

Property

The Euclidean space $\mathbb{R}^N$ is a <u>normed space</u>

It has an associated norm

$\|\cdot\| : \mathbb{R}^N \longrightarrow \mathbb{R}$

$i=1$

satisfying:

a) $\forall x \in \mathbb{R}^N \quad \|x\| > 0$ if $x \neq 0$

$\|x\| = 0$ if $x = 0$

b) $\|\lambda x\| = |\lambda| \, \|x\| \; , \; \forall \lambda \in \mathbb{R} , \; \forall x \in \mathbb{R}^N$

c) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$

Remark

Associated with the norm there exists the function "distance between two elements"

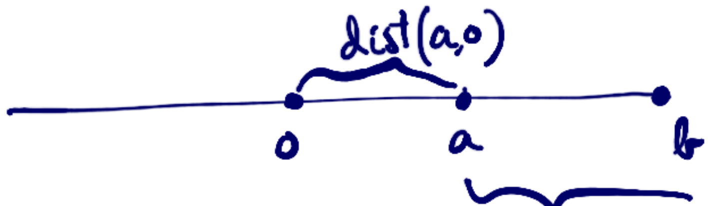
$$dist(x,y) = \|x-y\| \; , \; dist : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R} .$$

• $dist(x,y) = \|x-y\| \geq 0$

$= dist(x,z) + dist(y,z)$

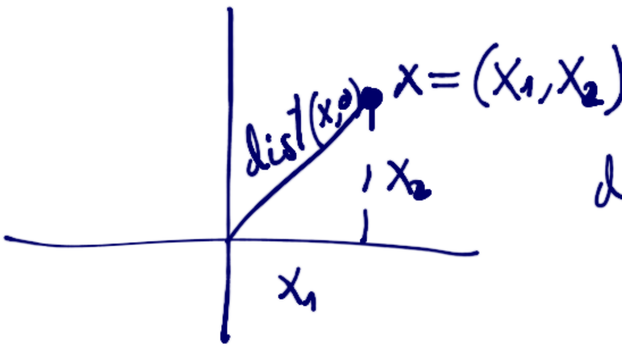
Remark.

In $\mathbb{R}$, dist ~ absolut value.



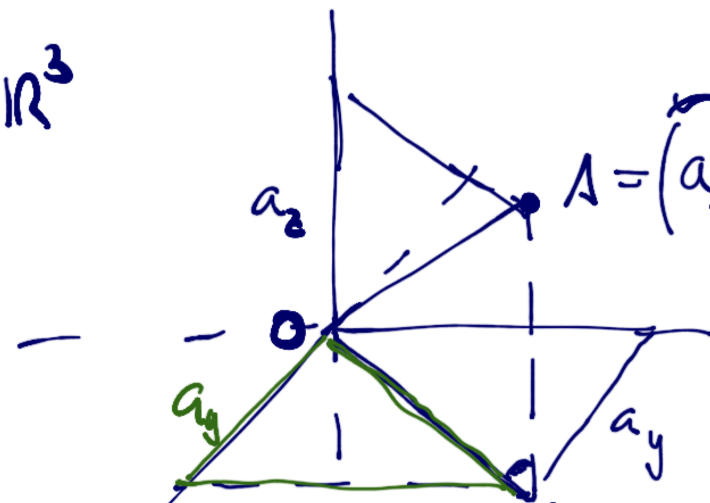
$$\mathrm{dist}(a,b)=|a-b|$$

In $\mathbb{R}^2$



$$\mathrm{dist}(x,0)=\|x\|=\sqrt{x_1^2+x_2^2}$$

In $\mathbb{R}^3$



$$\overbrace{\qquad}^{a}\quad A=(a_x,a_y,a_z)$$

$$OA=\mathrm{dist}(a,0)=\|a\|$$

$(or)=a_x+a_y$

Definition - Inner or scalar product

Let $a, b \in \mathbb{R}^N$ be two vectors in $\mathbb{R}^N$. Then the inner product is given by.

$$a \cdot b = a_1 b_1 + \dots + a_N b_N$$

$$a \cdot b = \langle a, b \rangle = (a, b)$$

Remark $\|x\| = \sqrt{\langle x, x \rangle} \equiv$ length of the vector x.

Properties

a) Positive definite $\langle x, x \rangle > 0$ if $x \neq 0$

Cauchy-Schwarz Inequality

$$|x \cdot y| \leq \|x\| \, \|y\|$$

Proof If $y = \lambda x$ (linearly dependent vectors)

$$|\langle x, \lambda x \rangle| = |\lambda \|x\|^2| = |\lambda| \, \|x\|^2 = \|x\| \, \|y\|$$

$$\|y\| = |\lambda| \, \|x\|$$

If $y \neq \lambda x$ (l.i vectors), we assume

$$z = \lambda x + y, \quad \lambda \in \mathbb{R}.$$

we know that

$$\langle \lambda x + y, \lambda x + y \rangle =$$

$$= \lambda^2 \|x\|^2 + 2\lambda \langle x, y \rangle + \|y\|^2$$

- Since y, x are l.i we cannot have real solutions for λ and then,

- $$4\left(\langle x,y\rangle\right)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\Downarrow$$

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$

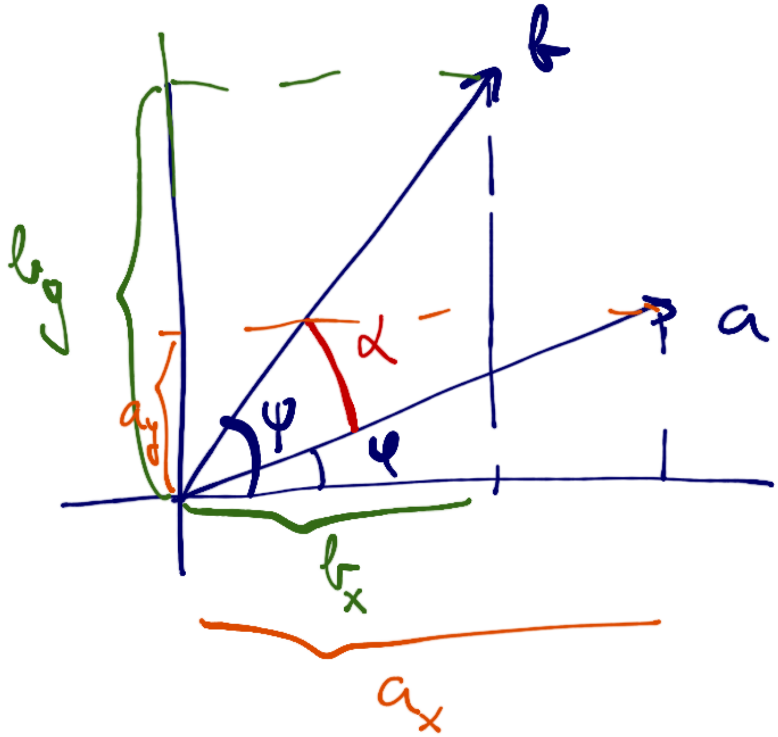
Geometrical interpretation for the scalar product

Theorem

$$\langle x,y\rangle = \|x\|\,\|y\|\cos\alpha$$

$\alpha \equiv$ angle between x and y.

Proof



$$a_x = \|a\| \cos\varphi \qquad a_y = \|a\| \sin\varphi$$

$$b_x = \|b\| \cos\psi \qquad b_y = \|b\| \sin\psi$$

$$\langle a, b \rangle = a_x b_x + a_y b_y = \|a\| \cos\varphi \, \|b\| \cos\psi + \|a\| \sin\varphi \, \|b\| \sin\psi$$

$$= \|a\| \, \|b\| \left(\cos\varphi \cos\psi + \sin\varphi \sin\psi \right)$$

$\|x\| \|y\|$

Examples $C(a,b)$ = continuous functions in $[a,b]$

$f,g \in C(a,b)$

$$\langle f,g \rangle = \int_a^b f(t)\,g(t)\,dt.$$

$$\langle x,y \rangle = x_1 y_1 + \cdots + x_N y_N$$

We might have a weight

$$\langle f,g \rangle = \int_a^b w(t)\, f(t)\, g(t)\, dt.$$

Such as

Families of orthogonal functions

Solutions for Heat equat., Wave Eq. . . . $\{\cos(nx), \sin(nx)\}$ in $[0, 2\pi]$

$$\int_0^{2\pi} \cos(nx)\sin(mx)\,dx = 0 = \int_0^{2\pi} \cos nx \cos mx\, dx \qquad n \neq m$$

$$0 = \int_0^{2\pi} \sin nx \sin mx\, dx$$

$$\|\cos nx\|^2 = \int_0^{2\pi} \cos^2 nx\, dx = \pi = \|\sin nx\|^2$$

Vector product (only $\mathbb{R}^3$)

$\boxed{x \times y \in \mathbb{R}^3}$

$x, y \in \mathbb{R}^3$ Notation.

$$\begin{vmatrix} i & j & k \\ \end{vmatrix} \quad \ldots \quad \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Geometric interpretation

To do so, let's start with the triple product

$$a\cdot(b\times c)=(a_1\, a_2\, a_3)\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} i-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} j+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix} k\right)$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} \Rightarrow \text{If } a \in \text{span}\{b, c\}$$

$$a\cdot(b\times c)=0$$

$b\times c$

Also

$$\boxed{\|b\times c\|^2} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}^2$$

$$= (b_2c_3 - c_2b_3)^2 + (b_1c_3 - c_1b_3)^2$$

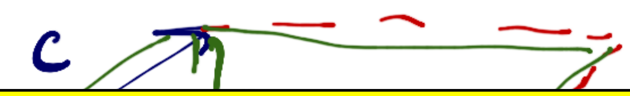
$$+ (b_1c_2 - c_1b_2)^2$$

$$= (b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2) - (\underline{b_1c_1 + b_2c_2 + b_3c_3})^2$$

$$= \|b\|^2\|c\|^2 - (b\cdot c)^2 = 1\|b\|^2\|c\|^2 - \|b\|^2\|c\|^2\cos^2\alpha$$

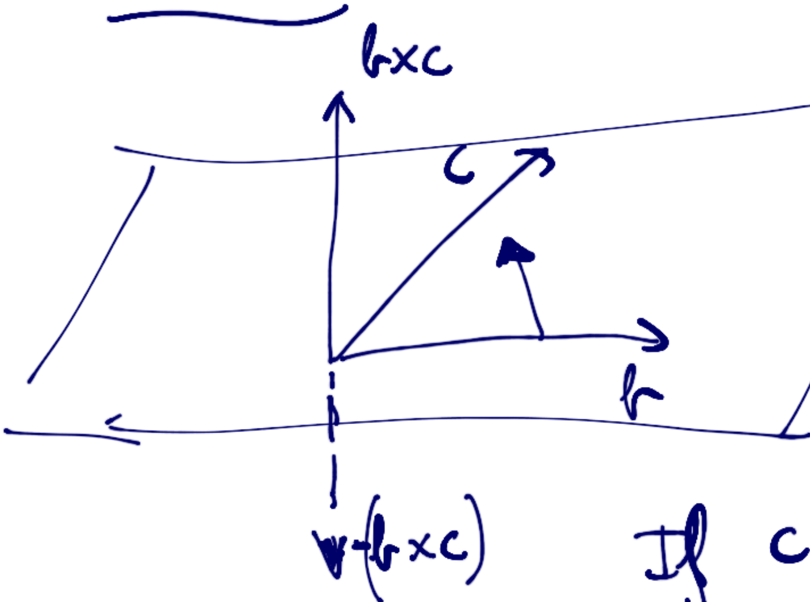
$$= \boxed{\|b\|^2\|c\|^2\sin^2\alpha}$$

c



$\|b\times c\| = \|b\|\,\|c\|\sin\alpha$ = area of parallelogram

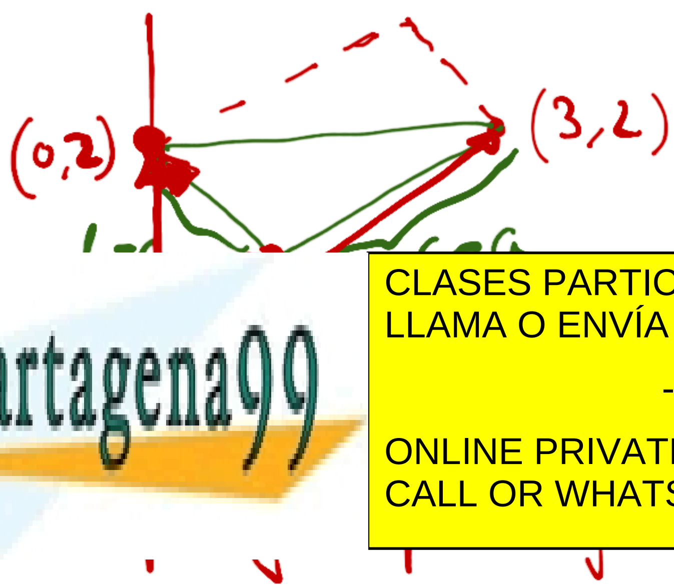
Remark



If $b \times c$ we choose the positive one.

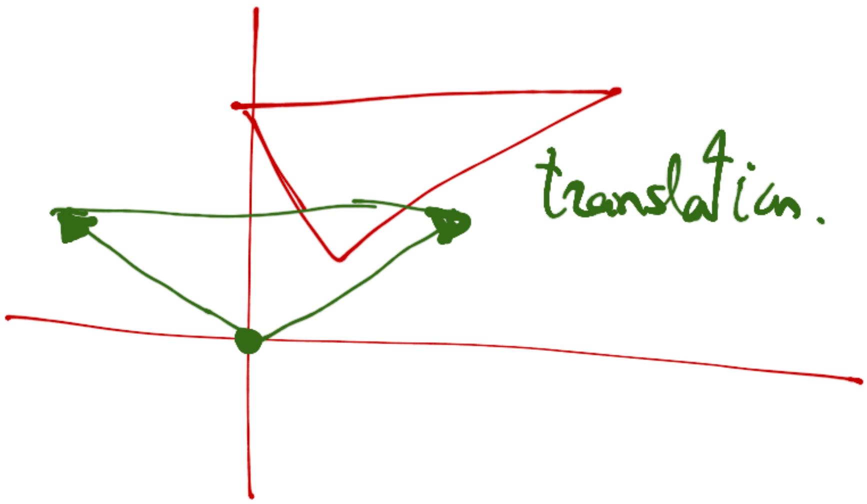
If $c \times b = -(b \times c)$ we choose the negative orthog. vector.

Example: Find the area of a triangle with vertexes on the points $(1,1)$, $(0,2)$ and $(3,2)$



Assume $\begin{cases} a = i + j \\ b = 2j \end{cases}$

$$\Delta = \frac{1}{2} \| (b-a) \times (c-a) \| \equiv \text{Area of the triangle.}$$



Topology of $\mathbb{R}^N$

Structure of open sets.

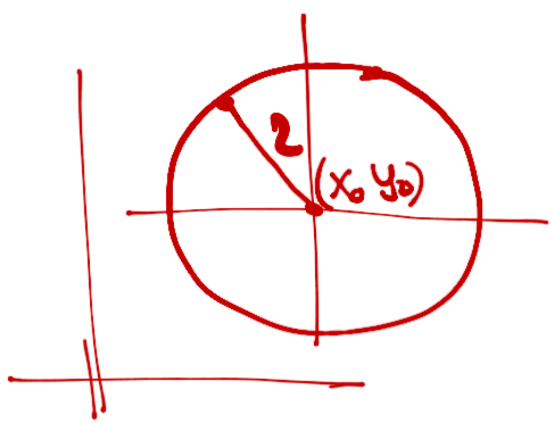
Open set in $\mathbb{R}^N$: We define an open ball centered at the point $x_0 \in \mathbb{R}^N$ with radius $r > 0$ as

$$\|X - X_0\| = \sqrt{(X_1 - X_{10})^2 + \cdots + (X_N - X_{N0})^2} < 2$$

$$(X_1 - X_{10})^2 + \cdots + (X_N - X_{N0})^2 < 2^2$$

Ex: In $\mathbb{R}^2$ formula for a circumference.

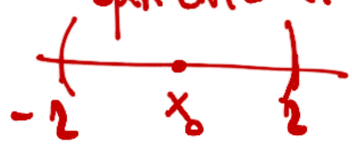
$$(x - x_0)^2 + (y - y_0)^2 = 2^2$$



Circle.

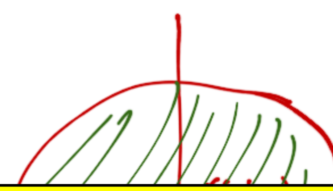
In $\mathbb{R}$

open interval.



$$\boxed{|x| < 2}$$

$$(x - x_0)^2 + (y - y_0)^2 < 2^2$$



Closed ball

$$\overline{B(x_0, r)} = \{ x \in \mathbb{R}^n ; \ dist(x, x_0) \leq r \}$$

Boundary of the ball

$$\partial B(x_0, r) = \{ x \in \mathbb{R}^n ; \ dist(x, x_0) = r \}$$

Example : $\mathbb{R}^3$